Abstract—In this letter, the problem of obtaining analytical bit error rate (BER) for multiple relays is addressed. This is done for the piecewise linear (PL) combiner. We obtain an exact computationally useful expression for similar fading parameters and an algorithm for the case when all parameters are different. Previously, exact expressions were available only for upto three relays. The novelty of this work lies in its practical utility. Analytical BER plots show the diversity gain due to multiple relays. The loss in diversity order, a theoretically well known result specific to the PL combiner, is now visible in the BER plots. These results are also verified through simulations.

Keywords—Cooperative diversity, decode and forward, piecewise linear

I. INTRODUCTION

In recent years, significant research has been done in cooperative diversity [1], [2], a practical alternative to classical MIMO. In particular, the performance analysis of decode and forward (DF) multirelay systems is of considerable interest. To the best of our knowledge, exact analysis for the performance of multirelay DF systems is rare [3]–[6].

The PL combiner is a practical alternative to the nonlinear maximum likelihood (ML) detector [1]. While other full diversity DF schemes have been proposed in the literature, the PL combiner is still relevant due to its simplicity, despite a loss in diversity order.

A. Previous Work

Exact BER analysis for the PL combiner is available in [3] for upto three relays between the source and destination. The BER expressions in [3] are huge and the analysis therein indicates that explicit exact expressions for more number of relays may not be feasible. [4] extends the approach in [3] for multiple relays, but does not propose any method to evaluate the residues, which is crucial for practical utility. For example, it has been theoretically shown in [1] that there is a loss in diversity order for multiple relays. But nowhere do the figures in [4] indicate this. Thus, for the analysis in [3], [4] to be practically useful, a method for computing the BER is necessary.

B. Contribution of this Work

In this letter, for the special case of similar fading parameters, we obtain an exact computationally useful expression for the BER for multiple relays. This expression can be used for determining the diversity gain for a given number of relays. We also design an algorithm to numerically evaluate the exact analytical BER for arbitrary fading parameters based on [3], [4], without explicitly evaluating the mathematical expression. Our analytical BER plots clearly show the loss in diversity order for the PL combiner for multiple relays. To the best of our knowledge, this has not been reported in the available literature.

II. SYSTEM MODEL AND PROBLEM DEFINITION

The multirelay system in [3], experiencing Rayleigh fading is considered, where the goal is to evaluate the BER

\[
P_r = \sum_x \prod_{i=1}^{N} e^{-\frac{x_i}{\delta}} (1 - e^{-\frac{x_i}{\delta}}) \frac{1}{\ln 2} P \left( X + \sum_{i=1}^{N} f(Y_i) < 0 | x = 1, x \right) \tag{1}
\]

Table I has details of the variables used in (1) and some notations in the rest of the paper. Binary phase shift keying (BPSK) is used for transmission and the fading is Rayleigh. More details are available in [3, Table I]. It is reiterated that \( X \) and \( Y_r \) have a similar distribution but different parameters. Their CF is given in Table II [3].

III. BER ANALYSIS: SIMILAR FADING

Letting \( V_r = f(Y_r) \) and noting that the variables \( X \) and \( Y_r \) in Table II are i.i.d., the conditional probability in (1) can be expressed as

\[
P \left( X + \sum_{i=1}^{N} V_r < 0 | x = 1, x \right) = \frac{1}{2\pi j} \oint_C \left[ M_X(t)M_{V_r}(t) \right] dt \tag{2}
\]

The authors are with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502205 India e-mail: praneeth-varma.gvss@gmail.com, gadeball@iith.ac.in.
TABLE II: CF of X and V = f(X).

<table>
<thead>
<tr>
<th>Variable</th>
<th>CF</th>
</tr>
</thead>
<tbody>
<tr>
<td>X, Y</td>
<td>( M_{x}(t) = \frac{1}{(t-a)^{m+1}} )</td>
</tr>
<tr>
<td></td>
<td>( M_{y}(t) = \frac{1}{(t-a)^{n+1}} )</td>
</tr>
</tbody>
</table>

From Table II,

\[
M_{v}^{N} = (1 - g_{x}(\alpha, t) - g_{y}(\beta, t))^{N} \quad (3)
\]

\[
= \sum_{m+n+k=N} \binom{N}{m, n, k} (-g_{x})^{m+n} \phi_{x}^{m}(\alpha, t) \phi_{y}^{n}(\beta, t) \quad (4)
\]

From Table I,

\[
\phi(\alpha, t) = \frac{t(1 - \delta^{n+1})}{\alpha(t - \alpha)} \quad (5)
\]

\[
\phi(\beta, t) = \frac{t(1 - \delta^{n+1})}{\beta(t - \beta)} \quad (6)
\]

Hence,

\[
\phi_{x}^{m}(\alpha, t) \phi_{y}^{n}(\beta, t) = \frac{t^{m+n}}{\alpha^{m} \beta^{n}} \frac{(1 - \delta^{n+1})}{\alpha(t - \alpha)} \frac{(1 - \delta^{n+1})}{\beta(t - \beta)} \quad (7)
\]

\[
= \frac{t^{m+n}}{\alpha^{m} \beta^{n}} \sum_{p=0}^{m} \sum_{q=0}^{n} (-1)^{p+q} \delta^{p, q, \alpha, \beta, (q-p)} \quad (8)
\]

Substituting (8) in (4),

\[
M_{v}^{N}(t) = \sum_{m+n+k=N} \binom{N}{m, n, k} (-g_{x})^{m+n} \frac{t^{m+n}}{\alpha^{m} \beta^{n}} \frac{(1 - \delta^{n+1})}{\alpha(t - \alpha)} \frac{(1 - \delta^{n+1})}{\beta(t - \beta)} \quad (9)
\]

Substituting (9) in (2),

\[
P \left( \sum_{r=1}^{N} V_{r} < 0 | x = 1, \lambda \right) = \frac{-g_{x}(\alpha + \beta)}{2\pi r} \sum_{m+n+k=N} \sum_{p=0}^{m} \sum_{q=0}^{n} \binom{N}{m, n, k} (-g_{x})^{m+n} \frac{t^{m+n}}{\alpha^{m} \beta^{n}} \frac{(1 - \delta^{n+1})}{\alpha(t - \alpha)} \frac{(1 - \delta^{n+1})}{\beta(t - \beta)} \times \int_{C} \frac{t^{m+n-1} \delta^{(q-p)t}}{(t - \alpha)(\beta + t)(t - \alpha)} \right) \quad (10)
\]

after simplification. Thus, the problem of finding a closed form expression for \( P_{e} \) in (1) reduces to the problem of solving the integral

\[
I = \int_{C} \frac{t^{m+n-1} \delta^{(q-p)t}}{(t - \alpha)(\beta + t)(t - \alpha)} \right) \quad (11)
\]

The integral in (11) can be evaluated in closed form using residue calculus [7]. The appropriate residues are evaluated in the following.

\section*{A. \( m = 0, n = 0 \)}

This is the only case when we encounter a pole at the origin. The corresponding residues are obtained as

\[
\text{Res}_{t \to 0} \frac{1}{(t - \alpha)(\beta + t)} = -\frac{1}{\alpha \beta} \quad (12)
\]

\[
\text{Res}_{t \to -\beta} \frac{1}{(t - \alpha)(\beta + t)} = \frac{1}{\beta(\alpha + \beta)} \quad (13)
\]

when poles in the left half plane, including 0 have been used for evaluating the residues.

\section*{B. \( p = q \)}

Here, the power of \( \delta \) vanishes. As a consequence, we can choose either half plane for evaluating the residue.

\section*{C. \( p < q \)}

In this case, the residues need to be evaluated in the left half plane. This yields

\[
\text{Res}_{t \to \beta} \frac{t^{m+n-1} \delta^{(q-p)t}}{(t - \alpha)(\beta + t)(t - \alpha)} = \frac{t^{m+n-1} \delta^{(q-p)t}}{(\alpha + \beta)(\beta + \alpha)(\beta - \beta)} \quad (14)
\]

From the above, the residues can be obtained as

\[
\text{Res}_{t \to -\beta} \frac{t^{m+n-1} \delta^{(q-p)t}}{(t - \alpha)(\beta + t)(t - \alpha)} \quad (15)
\]

Let

\[
G(t) = \ln \left[ \frac{t^{m+n-1} \delta^{(q-p)t}}{(t - \alpha)(\beta + t)(t - \alpha)} \right] \quad (16)
\]

\[
= (m + n - 1) \ln t + (q - p) \ln \delta - \ln(t - \alpha) - \ln(\beta + t) - n \ln(\alpha + \beta) \quad (17)
\]

We have,

\[
G^{(1)}(t) = \frac{m + n - 1}{t} \right] = \frac{1}{t - \alpha} - \frac{1}{t - \alpha} \quad (18)
\]

\[
G^{(2)}(t) = \frac{1}{(t - \alpha)^{2}} \left[ \frac{m + n - 1}{t} + \frac{1}{t - \alpha} \right] \quad (19)
\]

Using the Faa Di Bruno formula [8], the residue for \( -\beta \), can be expressed as

\[
\frac{1}{(n - 1)!} \sum_{i=1}^{n-1} \left[ (-\beta)^{i} \frac{m+n-1}{t^{i}} + \frac{1}{t - \alpha} \right] \frac{1}{(t - \alpha)^{i}} \quad (19)
\]

\[ \times B_{n-1,i} \left[ G^{(i)}(-\beta), G^{(2)}(-\beta), \ldots, G^{(n-1)}(-\beta) \right] \]

where \( B_{n-1,i} \) is the Bernoulli number of order \( n-1, i \).
D. $p > q$

In this case, residues need to be evaluated for poles in the right half plane. Also, a negative sign appears before the residue. The residues are

\[ \text{Res}_{z = \alpha} \left( \frac{m+n-1 \delta(q-p)\alpha}{(t-\alpha)(\beta+t)(t-\alpha)^n(t+\beta)^p} \right) = \frac{m+n-1 \delta(q-p)\alpha}{(\alpha+\beta)(\alpha-\alpha_r)^n(\alpha+\beta_r)^p} \]

and

\[ \text{Res}_{z = \alpha} \left( \frac{m+n-1 \delta(q-p)\alpha}{(t-\alpha)(\beta+\bar{\alpha})(\beta+\bar{\alpha})^n(t+\beta)^p} \right) = -\frac{1}{(m-1)!} \sum_{\alpha = 1}^{m-1} \left( (\alpha - \alpha)(\alpha + \beta)(\alpha + \beta)^n \right) \]

\[ \times B_{m-1,1} \left( H(1)(\alpha), H(2)(\alpha), \ldots, H(m-1)(\alpha) \right) \]

where

\[ H(t) = \ln \left( \frac{m+n-1 \delta(q-p)\alpha}{(t-\alpha)(\beta+t)(t+\beta)^p} \right) \]

IV. BER Analysis

Letting $V_r = f(Y_r)$ and noting that the variables $X$ and $Y_r$ in Table II are independent, (1) can be expressed as [9]

\[ P \left( X + \sum_{r=1}^{N} V_r < 0 \mid x_r = 1, x \right) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{0} \Phi_X(\omega) \prod_{r=1}^{N} \Phi_{V_r}(\omega) d\omega \]

\[ = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{0} \Phi_X(\omega) d\omega + \sum_{i} Q_k \]

where $|\cdot|$ represents the cardinality of a set and

\[ Q_k = \frac{1}{2\pi} \int_{-\infty}^{0} \frac{g(\alpha+\beta)}{\prod_{i=1}^{k} (\alpha + \beta_i)(\beta - \beta_i)} d\omega \]

A. Algorithm for BER Computation

This algorithm computes the integral in (25), by evaluating the residues for different poles. This integral has $\text{num_term}$, $\text{den_factor}$ and $\text{combination_factor}$ factors dependent upon $\alpha$. The evaluation of these terms is the key to the algorithm. In the following, we explain key steps of Algorithm 1 for $N = 3$.

1) In step 3, the various partitions $I_k, J_k$ of $Z_n$ need to be evaluated. For $N = 3, Z_3 = \{1, 2, 3\}$. $\sum_{k \in \mathcal{I}} \sum_{j : \mathcal{J} \subset k} Q_k$. The partitions of $Z_3$ required for the summation are

\[ \{I_k, J_k\}_{k=1}^{26} \]

2) Steps 5-9 are explained as follows. The subscript $i, j \in I_k$ is used for the pole in upper half i.e. $-\gamma_i$. The subscript $j, l \in J_k$ is used for the pole in lower half i.e. $\beta_j$. Each combination results in different $\delta$ multiplications, in $\mathcal{N}$, for example, $N=3, k=21$ in the example corresponds to $\{1, 2, 3\} = \{|1, 2, 3\}$ which in turn corresponds to an integral involving $\{|\alpha_1, \alpha_2, \beta_1\}$. For one of the integrands in this integral, $\gamma_1$ is not present. This combination may be referred to as $[0, 1, 1]$. Then we obtain a term $\frac{1}{\pi}$ and define $\text{sum_num}$ as $\delta_2^{\alpha_1+\text{pole}} * \delta_3^{\alpha_2-\text{pole}}$.

3) Steps 8-12 are related to evaluation of the $\text{num_term}$ in (25). Considering $Q_{21}$, from (25), after some algebra,

\[ \text{num_term} = \prod_{i \in \{1, 2\}} \prod_{j \in \{3\}} \left( 1 - \delta_i^{\alpha_j + z} \right) \left( 1 - \delta_j^{\beta_i - z} \right) \]

\[ = \left( 1 - \delta_1^{\alpha_1} \delta_1^{z} \right) \left( 1 - \delta_2^{\alpha_2} \delta_2^{z} \right) \left( 1 - \delta_3^{\beta_1} \delta_3^{z} \right) \]

\[ + \delta_1^{\alpha_1} \delta_3^{\beta_1} \left( 1 - \delta_1^{\alpha_2} \delta_3^{z} \right) \left( 1 - \delta_2^{\alpha_2} \delta_3^{z} \right) \]

Thus, the integral in (25) is a sum of several other integrals. Each of these integrals can be evaluated using residue calculus on the condition whether the radices of the expressions on the left column of Table III are less than or greater than 1, corresponding to the poles in lower and upper half planes respectively [3]. These are listed assuming $\delta_1 < \delta_2 < \delta_3$. For example, $\text{num_term}$ for the poles $z = \beta, -\alpha$ are obtained below using Table III and (26) are

\[ \begin{align*}
1 - \delta_1^{\alpha_1 + \beta} + \delta_2^{\alpha_2 + \beta} + \delta_3^{\alpha_3 + \beta} \\
+ \delta_1^{\beta_1} \delta_2^{\alpha_2} \delta_3^{\alpha_3} & z = \beta, \text{lower half} \\
+ \delta_1^{\alpha_1} \delta_3^{\beta_1} \delta_2^{\alpha_2} & z = -\alpha, \text{upper half}
\end{align*} \]
**Algorithm 1** The multirelay PL-DF algorithm

1: **procedure** BER(N, δ, α, β)
2: \( \text{all\_comb} = 0 \)
3: **for all** \( k \in \{I_k, J_k\} \) **do**
4: \( \text{each\_comb} = 0 \)
5: **for all** \( i \in I_k \cup J_k \cup \{-\alpha, \beta\} \) **do**
6: \( \text{num\_term} = 0 \)
7: **for all combinations \( \in 2^{|I_k|+|J_k|} \) **do**
8: **if** \( i \in (J_k \cup \{\beta\}) \) **and condition \( \leq 1 \) **then**
9: evaluate \( \text{sum\_num} \)
10: **else if** \( i \in (I_k \cup \{-\alpha\}) \) **and condition \( > 1 \) **then**
11: evaluate \( \text{sum\_num} \)
12: **end if**
13: \( \text{den\_term} = \text{num\_term} + \text{sum\_num} \)
14: **end for**
15: \( \text{den\_factor} = 1 \)
16: **for all** \( l \in I_k \cup J_k \cup \{-\alpha, \beta\} \) **do**
17: **if** \( l \neq i \) **then**
18: \( \text{den\_factor} = \text{den\_factor} \ast \left(\frac{1}{1-\alpha}\right)\)
19: **end if**
20: **end for**
21: \( \text{pole\_factor} = (|J_k|+|I_k|)^{-1} \)
22: \( \text{each\_pole} = \text{den\_factor} \ast \text{pole\_factor} \)
23: \( \text{all\_pole} = \text{all\_pole} + \text{each\_pole} \)
24: **end for**
25: \( \text{combination\_factor} = 1 \)
26: **for all** \( l \in I_k \cup J_k \) **do**
27: \( \text{combination\_factor} = \text{combination\_factor} \ast \left(\frac{\delta}{I}\right)\)
28: **end for**
29: \( \text{each\_comb} = \text{all\_pole} \ast \text{combination\_factor} \)
30: \( \text{all\_comb} = \text{all\_comb} + (-1)^{|I_k|+|J_k|-1} \ast \text{each\_comb} \)
31: **end for**
32: \( P \left( X + \sum_{r=1}^{N} V_r < 0 | x_k = 1, x \right) = \frac{\delta}{\pi} + g(\alpha + \beta) \ast \text{all\_comb} \)
33: **end procedure**

was theoretically shown in [1] but is now confirmed in Figure 1. The fall in the BER with increasing number of relays establishes relay diversity through PL-DF. In Figure 2, BER plots are shown for increasing number of relays at different locations. This demonstrates the generality of our result.

VI. **Conclusions**

In this paper, we have proposed an algorithm for evaluating the average BER for arbitrary number of relays for a PL-DF system, based on the ideas in [3]. Numerical results confirm that the PL combiner suffers from a loss in diversity order. Clearly, there is a need for simple cooperative techniques that yield full diversity. Existing techniques offer only one of these features. This problem needs to be addressed in future work.
Fig. 2: Analytical results for the PL-DF for $N = 1, 2, \ldots, 5$ relays (top to bottom) located at $L = \{.3, .5, .7, .4, .6\}$.

REFERENCES


